Exercise 51

The cost (in dollars) of producing x units of a certain commodity is $C(x) = 5000 + 10x + 0.05x^2$.

(a) Find the average rate of change of C with respect to x when the production level is changed

(i) from $x = 100$ to $x = 105$

(ii) from $x = 100$ to $x = 101$

(b) Find the instantaneous rate of change of C with respect to x when $x = 100$. (This is called the *marginal cost*. Its significance will be explained in Section 3.7.)

Solution

The average rate of change of C with respect to x from $x = 100$ to $x = 105$ is

$$\frac{C(105) - C(100)}{105 - 100} = \frac{[5000 + 10(105) + 0.05(105)^2] - [5000 + 10(100) + 0.05(100)^2]}{5} = 20.25\ \frac{\text{dollars}}{\text{unit}},$$

and the average rate of change of C with respect to x from $x = 100$ to $x = 101$ is

$$\frac{C(101) - C(100)}{101 - 100} = \frac{[5000 + 10(101) + 0.05(101)^2] - [5000 + 10(100) + 0.05(100)^2]}{1} = 20.05\ \frac{\text{dollars}}{\text{unit}}.$$

The instantaneous rate of change of C is the derivative of C.

$$\begin{aligned}
C'(x) &= \lim_{h \to 0} \frac{C(x+h) - C(x)}{h} \\
&= \lim_{h \to 0} \frac{[5000 + 10(x+h) + 0.05(x+h)^2] - [5000 + 10x + 0.05x^2]}{h} \\
&= \lim_{h \to 0} \frac{[5000 + 10x + 10h + 0.05(x^2 + 2xh + h^2)] - 5000 - 10x - 0.05x^2}{h} \\
&= \lim_{h \to 0} \frac{(5000 + 10x + 10h + 0.05x^2 + 0.1xh + 0.05h^2) - 5000 - 10x - 0.05x^2}{h} \\
&= \lim_{h \to 0} \frac{10h + 0.1xh + 0.05h^2}{h} \\
&= \lim_{h \to 0} (10 + 0.1x + 0.05h) \\
&= 10 + 0.1x
\end{aligned}$$

Therefore, the instantaneous rate of change of C with respect to x when $x = 100$ is

$$C'(100) = 10 + 0.1(100) = 20\ \frac{\text{dollars}}{\text{unit}}.$$